$\cdots$), the Nth set corresponding to values of n_1 and n_2 such that their sum has the fixed value N. Thus the Nth set of equations is, explicitly,

$$i\{[(s+1)(N-s)]^{\frac{1}{2}}\langle s+1\ N-s-1\ n_3|l_3'\rangle - [s(N-s+1)]^{\frac{1}{2}}\langle s-1\ N-s+1\ n_3|l_3'\rangle = l_3'\langle s\ N-s\ n_3|l_3'\rangle, \quad (s=0,1,\cdots,N). \quad (3.1)$$

Inspection reveals at once that the eigenvalues of l_3 are those of the matrices $A^{(N)}$ ($N=0,1,2,\cdots$), the elements of which are

$$A_{st}^{(N)} = i\{[(s+1)(N-s)]^{\frac{1}{2}}\delta_{s,t-1} - [s(N-s+1)]^{\frac{1}{2}}\delta_{s,t+1}\}. \quad (3.2)$$

Now if $\mathbf{J}$ is a general angular momentum, one has the following well-known results (e.g., reference 5, pp. 339–345): (i) The eigenvalues of J_z (and therefore of J_x and J_y) are half-integral, i.e., $J_z'=0, \pm\frac{1}{2}, \pm 1, \cdots$); (ii) in the J^2, J_z-diagonal representation (in which J_x+iJ_y acts as a raising operator) the representative of J_y is a diagonal block matrix such that if its j, j element (corresponding to $J^{2\prime}=j(j+1)$, $j=0, \frac{1}{2}, 1, \cdots$) be denoted by $J_y^{(j)}$, then $J_y^{(j)}$ is a $(2j+1)$-dimensional matrix whose elements are

$$(J_y^{(j)})_{m''m'} = \tfrac{1}{2}i\{[(j+m')(j-m'+1)]^{\frac{1}{2}}\delta_{m'',m'-1} - [(j-m')(j+m'+1))^{\frac{1}{2}}\delta_{m'',m'+1}\}. \quad (3.3)$$

Here m', m'' run by integral steps from j to $-j$. The relabelling defined by

$$m'=j-t, \quad m''=j-s \quad (s,t=0,1,\cdots,2j)$$

changes (3.3) into

$$(J_y^{(j)})_{st} = -\tfrac{1}{2}i\{[(s+1)(2j-s)]^{\frac{1}{2}}\delta_{s,t-1} - [s(2j-s+1)]^{\frac{1}{2}}\delta_{s,t+1}\}. \quad (3.4)$$

Comparison of (3.2) with (3.4) now shows that

$$A^{(N)} = -2J_y^{(\frac{1}{2}N)}. \quad (3.5)$$

However, as already remarked, the eigenvalues of $J_y^{(j)}$ are half-integral, so that those of $A^{(N)}$ are integral. Thus *all the eigenvalues of l_3 are integral* (including zero), which was to be shown. Moreover, the general theory of angular momentum then leads to the conclusion that the total orbital angular momentum quantum number l must also be integral.

Compressible Fluid Flow and the Theory of Characteristics*

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This paper calls attention to the contemporary need for an elementary discussion of the nonlinear features of the equations of hydrodynamics and their application to supersonic and high pressure phenomena. A simplified account of the method of characteristics is presented, with illustrations from the theory of the ideal fluid, the thermally conducting fluid, and the viscous fluid. An "asymptotic paradox" is briefly discussed.

INTRODUCTION

QUITE frequently the fluid mechanics problems, encountered by physicists in research and development activities, involve flow velocities and compressions, such that the linear approximations to the flow equations cannot be usefully employed. The solution of even the simplest of fluid flow problems then becomes a problem in nonlinear partial differential equations.

Since the Second World War, several books have been published concerning nonlinear partial differential equations as related to shock wave, detonation, and supersonic flow problems.[1–4] A

* Supported by the National Science Foundation Grant NSF G6556.

[1] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948).
[2] R. von Mises, *Mathematical Theory of Compressible Fluid Flow* (Academic Press, Inc., New York, 1958).
[3] A. H. Shapiro, *The Dynamics and Thermodynamics of Compressible Fluid Flow* (Roland Press Company, New York, 1958).
[4] R. H. Cole, *Underwater Explosions* (Princeton University Press, Princeton, New Jersey, 1948).

young physicist starting research in such fields must often familiarize himself with the fundamentals by independent study, because the subject of fluid mechanics is not generally part of a physics curriculum. This paper is primarily intended as an aid to such a person, particularly with reference to the work by Courant and Friedrichs,[1] and the more recently published work by von Mises.[2] Unfortunately, the treatment and notation of the first reference are such as to be intelligible only to mathematical physicists of considerable maturity and sophistication. However, the second reference develops the concepts of the characteristics of nonlinear differential equations in a manner that is both intuitively more transparent and mathematically more general than previous disucssions. The von Mises text is, of course, written at an advanced level, and the present authors believe that its approach is so clear and so important that it is worthwhile to attempt a somewhat simplified version suitable for self-study by graduate students in physics. This paper is a step in this direction. It will be assumed that the reader is familiar with the elementary discussions such as those usually included in a physics degree program,[5,6] in particular the Cauchy problem, and with the concepts of vector analysis as presented in standard courses on mechanics or theoretical physics.[7] The von Mises device of using a vector notation in the $x-t$ plane is an excellent and convenient shorthand way of writing the partial differential equations that is independent of the coordinate system used, and that readily lends itself to the discussion of characteristics.

In particular the authors feel that writing the gradient operator as the vector sum of a normal and a tangential derivative along the curve of initial data [Eq. (15)] helps clarify the meaning of the characteristics. From this approach, the linear combination of equations to form a directional derivative in the characteristic direction, somewhat arbitrarily introduced by Courant and Friedrichs in their Section 22, appears quite naturally as von Mises' compatability relations. The Riemann invariants, to be used along the characteristic curves to solve the flow problem, are then obtained from the compatability relations.

As an illustration of the power of the von Mises approach, the more general case of a heat-conducting fluid is discussed. The equations governing this case are second order, and consequently the vector notation must be extended to a tensor or dyadic notation. In this notation the distinction between the nature of the flow of a fluid with exceedingly small thermal conductivity and that of an ideal fluid with zero thermal conductivity is immediately apparent.

It is also hoped that this paper may help stimulate teachers responsible for the planning of graduate programs in physics to include more reading assignments in nonlinear mechanics and the theory of characteristics.

EQUATIONS OF COMPRESSIBLE FLUID FLOW

We shall first consider only the simplest possible model, a compressible thermodynamically reversible (nonviscous, nonthermally conducting, etc.,) ideal fluid, and at first also restrict the discussion to motion in only one direction, taken as the x axis. Let the density and pressure of the fluid be ρ and p, the "particle velocity" v, at position x and time t. The equations governing the motion of the fluid are then the equation of mass conservation

$$\partial\rho/\partial t+\partial(\rho v)/\partial x=0, \qquad (1)$$

and momentum change

$$\rho(\partial v/\partial t+v\partial v/\partial x)+\partial p/\partial x=0. \qquad (2)$$

An equation expressing the pressure as a function of the density must also be supplied to specify the material properties of the fluid—the specifying equation

$$F(p,\rho)=0. \qquad (3)$$

Given the specifying equation we can determine

$$c^2=dp/d\rho, \qquad (4)$$

where c is the velocity of propagation of small amplitude sound waves. We use this to rewrite the second equation of motion in terms of ρ and v, and then set out the two equations of motion

[5] A. Sommerfeld, *Partial Differential Equations* (Academic Press, Inc., New York, 1949).

[6] D. Greenspan, *Introduction to Partial Differential Equations* (McGraw-Hill Book Company, Inc., New York, 1961).

[7] W. Band, *Introduction to Mathematical Physics* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1959).

in the following pattern:

$$\partial\rho/\partial t+v\partial\rho/\partial x+0+\rho\partial v/\partial x=0, \quad (5a)$$

$$0+c^2\partial\rho/\partial x+\rho\partial v/\partial t+\rho v\partial v/\partial x=0. \quad (5b)$$

This pair of equations can then be written in a vector notation

$$\sum_k \mathbf{a}_{ik}\cdot\mathbf{D}u_k=0, \quad i,k=0,1. \quad (6)$$

Here the vector operator $\mathbf{D}$ is the pair of differentials $(\partial/\partial t,\partial/\partial x)$, while $u_0=\rho$, $u_1=v$; and if the subscript $i=0$ refers to the first of Eqs. (5), and $i=1$ to the second, then the vectors $\mathbf{a}_{ik}$ are:

$$\mathbf{a}_{00}=(1,v);\quad \mathbf{a}_{01}=(0,\rho);\quad \mathbf{a}_{10}=(0,c^2);\quad \mathbf{a}_{11}=(\rho,\rho v). \quad (7)$$

The equations of motion in three dimensions are

$$\left.\begin{aligned}\partial\rho/\partial t+\mathbf{v}\cdot\mathrm{grad}\rho+\rho\,\mathrm{div}\,\mathbf{v}=0\\ c^2\,\mathrm{grad}\rho+\rho\partial\mathbf{v}/\partial t+\rho\mathbf{v}\cdot\mathrm{grad}\mathbf{v}=0\end{aligned}\right\}, \quad (8)$$

and these four equations can be written in exactly the same form as (6) if we take i, k ranging from 0 to 3, and the notation

$$\mathbf{D}=(\partial/\partial t,\partial/\partial x,\partial/\partial y,\partial/\partial z), \quad (9)$$

$u_0=\rho$, $u_1=v_x$, $u_2=v_y$, $u_3=v_z$, and the sixteen vectors

$$\left.\begin{aligned}&\mathbf{a}_{00}=(1,u_1,u_2,u_3);\quad \mathbf{a}_{01}=(0,\rho,0,0);\\ &\qquad \mathbf{a}_{02}=(0,0,\rho,0);\quad \mathbf{a}_{03}=(0,0,0,\rho),\\ &\mathbf{a}_{10}=(0,c^2,0,0);\quad \mathbf{a}_{11}=(\rho,\rho u_1,\rho u_2,\rho u_3);\\ &\qquad \mathbf{a}_{12}=\mathbf{a}_{13}=0,\\ &\mathbf{a}_{20}=(0,0,c^2,0);\quad \mathbf{a}_{21}=\mathbf{a}_{23}=0;\\ &\qquad \mathbf{a}_{22}=(\rho,\rho u_1,\rho u_2,\rho u_3),\\ &\mathbf{a}_{30}=(0,0,0,c^2);\quad \mathbf{a}_{31}=\mathbf{a}_{32}=0;\\ &\qquad \mathbf{a}_{33}=(\rho,\rho u_1,\rho u_2,\rho u_3).\end{aligned}\right\} \quad (10)$$

In general, we are going to study systems of differential equations having the form

$$\sum_k \mathbf{a}_{ik}\cdot\mathbf{D}u_k=b_i \quad i,k=0,1,\cdots,N, \quad (11)$$

where each vector $\mathbf{a}$ is a function of all the variables u_k, and the quantities b_k are also functions of the variables u_k. The number of components in the vectors is equal to the number of independent variables in the problem, e.g., four in Eq. (8), and need not be equal to N, which is the number of dependent variables in the problem.

CHARACTERISTICS WITH TWO INDEPENDENT VARIABLES

We return now to the simple problem presented by Eq. (6), slightly generalized to read

$$\sum_k \mathbf{a}_{ik}\cdot\mathbf{D}u_k=b_i \quad i,k=0,1, \quad (12)$$

where each component of the four vectors $\mathbf{a}_{ik}$ and both the quantities b_i are given explicit functions of ρ and v, (i.e., of u_0 and u_1). From the physics of the situation we would expect that a unique solution, giving ρ and v as functions of x at all times t, exists if the "initial values" of ρ and v are given at $t=0$ for all positions x; more generally, if ρ and v are given at all points on some specified "initial curve" in the $x-t$ plane, then ρ and v are uniquely determined throughout the $x-t$ plane. It turns out, however, that not all "initial curves" will give such predictive information, and for a very clear physical reason to be explained later. These exceptional initial curves are the "characteristics" of the set of differential equations.

The general method of solving equations of the type (11) and (12) that we are going to study involves first locating the characteristic curves.

Let an arbitrary initial curve C in the $x-t$ plane be specified parametrically by expressing t and x, or x_0 and x_1 as functions of a parameter σ.

$$x_0=x_0(\sigma),\quad x_1=x_1(\sigma), \quad (13)$$

and let the values of the dependent variables prescribed along the curve C also be expressed in terms of the same parameter on C:

$$u_k=u_k[x_0(\sigma),x_1(\sigma)]. \quad (14)$$

Let $\boldsymbol{\tau}$ be a unit vector tangent to C at some point, and $\boldsymbol{\lambda}$ be a unit vector normal to C at the same point, in the $x-t$ plane. The vectors $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ form a basis, and the vector operator $\mathbf{D}$ can be written in the form

$$\mathbf{D}=\boldsymbol{\tau}\partial/\partial\sigma+\boldsymbol{\lambda}\partial/\partial n, \quad (15)$$

where $\partial/\partial\sigma$ is a directional derivative tangential, and $\partial/\partial n$ a directional derivative normal to the curve C. The set of equations (12) can now be written in the form

$$\sum_k \mathbf{a}_{ik}\cdot(\boldsymbol{\tau}\partial u_k/\partial\sigma+\boldsymbol{\lambda}\partial u_k/\partial n)=b_i. \quad (16)$$

Now, by hypothesis, both the functions u_k are known all along C, so if we know $\mathbf{D}u_k$ all along C, we can construct the values of u_k in the neighborhood of C. Knowledge of u_k along C at once yields $\partial u_k/\partial\sigma$, but we still need $\partial u_k/\partial n$ to complete the solution. We can rewrite (16) as

$$\sum_k (\mathbf{a}_{ik}\cdot\boldsymbol{\lambda})\partial u_k/\partial n = b_i - \sum_k (\mathbf{a}_{ik}\cdot\boldsymbol{\tau})\partial u_k/\partial\sigma. \quad (17)$$

All quantities on the right, and all the coefficients $(\mathbf{a}_{ik}\cdot\boldsymbol{\lambda})$ on the left are known along C, and so Eq. (17) can be regarded as a pair of equations for the two derivatives $\partial u_k/\partial n$ at every point on C. The solution of this pair of equations, if one exists, will provide the information needed to construct u_k along any curve lying near to C. Repeating the process indefinitely will yield the complete solution to our problem.

The condition that a solution of Eqs. (17) does exist is that the determinant of the coefficients on the left side shall not vanish. But we are interested in curves C for which solutions of Eq. (17) do not exist, and this occurs if the determinant of the coefficients does vanish

$$\begin{vmatrix} \mathbf{a}_{00}\cdot\boldsymbol{\lambda} & \mathbf{a}_{01}\cdot\boldsymbol{\lambda} \\ \mathbf{a}_{10}\cdot\boldsymbol{\lambda} & \mathbf{a}_{11}\cdot\boldsymbol{\lambda} \end{vmatrix} = 0. \quad (18)$$

The unit vector $\boldsymbol{\lambda}$ determined by this equation as a function of position in the $x-t$ plane is everywhere normal to the curve C which is a characteristic of the equations of motion. Data along such a curve cannot be used to derive a solution of the equations. Equation (18) gives $\boldsymbol{\lambda}$ as a field in the $x-t$ plane, and this field enables us to construct the entire family of characteristics. Since there are, in general, two roots of Eq. (18), there are two families of characteristics.

If we write

$$\boldsymbol{\lambda} = (\lambda_0, \lambda_1), \quad (19)$$

and use Eq. (7) in Eq. (18), we find the two roots to be

$$\lambda_0/\lambda_1 = -v \pm c. \quad (20)$$

The slopes of the two characteristic curves, normal to the vector $\boldsymbol{\lambda}$, are therefore given by

$$\tau_1/\tau_0 = c+v \quad \text{or} \quad -(c-v). \quad (21)$$

These are "signal velocities" in the fluid, and it is for this reason, physically, that such characteristics cannot be used for predictive purposes. In the first place, if one could set up an arbitrary set of values of ρ and v on such a line, the information so presented would propagate with the signal velocity and so remain on the line; and in the second place, since information travels along the line, it is in fact not possible to set up an arbitrary choice of ρ and v on the line: one's choice at a point A will already determine what one must choose at a neighboring point. There exist in fact "compatibility" relations on each characteristic. We proceed to derive these.

THE COMPATIBILITY RELATIONS

When $\boldsymbol{\lambda}$ is normal to a characteristic curve C, the vanishing of the determinant Eq. (18) means algebraically that the two quantities on the left of Eq. (17), namely $\sum_k (\mathbf{a}_{0k}\cdot\boldsymbol{\lambda})\partial u_k/\partial n$ and $\sum_k (\mathbf{a}_{1k}\cdot\boldsymbol{\lambda})\partial u_k/\partial n$, are not independent—there exists a linear combination that vanishes. In other words, there exists a pair of coefficients α_i such that

$$\sum_i \alpha_i \sum_k (a_{ik}\cdot\lambda)\partial u_k/\partial n = 0, \quad (22)$$

while Eq. (17) then implies that

$$B = \sum_i \alpha_i b_i = \sum_i \alpha_i \sum_k (a_{ik}\cdot\tau)\partial u_k/\partial\sigma. \quad (23)$$

Now since the values of u_k on the characteristic do not determine $\partial u_k/\partial n$, the normal derivative remains arbitrary across the characteristic, and therefore Eq. (22) can be true only if

$$\sum_i \alpha_i \mathbf{a}_{ik}\cdot\boldsymbol{\lambda} = 0 \quad \text{for each } k. \quad (24)$$

In other words, the vectors

$$\mathbf{A}_k = \sum_i \alpha_i \mathbf{a}_{ik} \quad k = 0, 1 \quad (25)$$

are both tangential to the characteristic curve C. Therefore,

$$\mathbf{A}_k\cdot\boldsymbol{\tau} = \pm|\mathbf{A}_k|, \quad (26)$$

and Eq. (23) becomes

$$\pm\sum_k |\mathbf{A}_k|\,\partial u_k/\partial\sigma = \pm\sum_k \mathbf{A}_k\cdot\mathbf{D}u_k = B. \quad (27)$$

As $|\mathbf{A}_k|$ and B are known at all points in the plane, this equation is a restriction on the tangential derivatives of the u's along the characteristics—it is a compatibility relation.

For each root λ of Eq. (18), i.e., for each characteristic direction, there is a set of coefficients α_i satisfying Eq. (24). There is thus a pair of vectors $\mathbf{A}_k$ for each characteristic, and so a compatibility relation along each characteristic direction. We shall proceed to find these compatibility relations for the characteristics found in Eq. (21).

First, we write out Eq. (24) in detail, using (7) for $\mathbf{a}_{ik}$ and (20) for λ.

$$\left.\begin{array}{l}\alpha_0(1,v)\cdot(-v\pm c,\ 1)\lambda_0\\ \qquad+\alpha_1(0,c^2)\cdot(-v\pm c,\ 1)\lambda_0=0\\ \alpha_0(0,\rho)\cdot(-v\pm c,\ 1)\lambda_0\\ \qquad+\alpha_1(\rho,\rho v)\cdot(-v\pm c,\ 1)\lambda_0=0\end{array}\right\}. \tag{28}$$

Both equations naturally yield the same answer:

$$\alpha_0/\alpha_1=\mp c, \tag{29}$$

the absolute magnitude of either factor being immaterial.

For the so-called "plus characteristics" given by $\tau_1/\tau_0=c+v$ and $\lambda_0/\lambda_1=-v-c$, the multipliers are $\alpha_0/\alpha_1=+c$, and we may take $\alpha_0=c$, $\alpha_1=1$. From Eq. (25) we now have

$$\left.\begin{array}{l}\mathbf{A}_0=c(1,v)+(0,c^2)=c(1,\ c+v)\\ \mathbf{A}_1=c(0,\rho)+(\rho,\rho v)=\rho(1,\ c+v)\end{array}\right\}, \tag{30}$$

and the compatibility relation (27) reduces to

$$c(1,\ c+v)\cdot(\partial/\partial t,\partial/\partial x)\rho+\rho(1,\ c+v)\cdot(\partial/\partial t,\partial/\partial x)v=0,$$

or

$$(c/\rho)[\partial/\partial t+(c+v)\partial/\partial x]\rho+[\partial/\partial t+(c+v)\partial/\partial x]v=0. \tag{31}$$

This has exactly the first form on the left of Eq. (27) with

$$\partial/\partial\sigma=\partial/\partial t+(c+v)\partial/\partial x, \tag{32}$$

which is the directional derivative along the characteristic, with σ being the signalling time on the characteristic. Finally, if we define the new variable $L(\rho)$ by

$$L(\rho)=\int_{\rho_0}^{\rho}c(\rho')d\ \ln\rho', \tag{33}$$

and note that

$$\partial L/\partial t=(c/\rho)\partial\rho/\partial t,\quad \partial L/\partial x=(c/\rho)\partial\rho/\partial x, \tag{34}$$

the compatibility relation (31) reduces simply to

$$(\partial/\partial\sigma)(v+L)=0. \tag{35}$$

Thus the compatibility relation asserts simply that the function $v+L$ is a constant along the characteristic; the function $v+L$ is known as the Riemann invariant for the characteristic.

For the other family of characteristics, given by $\tau_1/\tau_0=-c+v$, we find in a similar fashion that

$$(\partial/\partial\sigma)(v-L)=0. \tag{36}$$

It is conventional to use the symbol σ_+ for the signalling time on the plus-characteristic, and σ_- for the same parameter on the minus-characteristic; the two compatibility relations are then written as

$$(\partial/\partial\sigma_+)(v+L)=0,\quad (\partial/\partial\sigma_-)(v-L)=0, \tag{37}$$

where

$$L(\rho)=\int_{\rho_0}^{\rho}c(\rho')d\ \ln\rho'.$$

Note that the characteristic directions are not just properties of the fluid, but are properties of the motion of the fluid, and depend on the initial conditions prescribed by the problem. Starting from the initial conditions one can build up the solution and the characteristics in a step-by-step process described in the next section.

INTEGRATION OF THE EQUATIONS OF MOTION

Let ρ and v be prescribed on a curve PQ in the $x-t$ plane, that is nowhere tangent to a characteristic direction; since we really do not know the characteristic directions before the prescription is set up, we must be sure that the prescription of ρ and v on PQ does not happen to force the characteristics to be tangent to PQ at any point. Knowing ρ and v is equivalent to knowing c and v at all points on PQ, and therefore knowing the slopes of the characteristics Eq. (21) as they cross PQ. Choose any two points A, B on PQ close enough together that $\rho_B-\rho_A\ll\rho_A$, and draw straight lines AC and BC in the directions of the plus- and minus-characteristics, respectively, intersecting at C. From the data on PQ we know ρ, and therefore $L(\rho)$ at both A and B, and we may write the compatibility

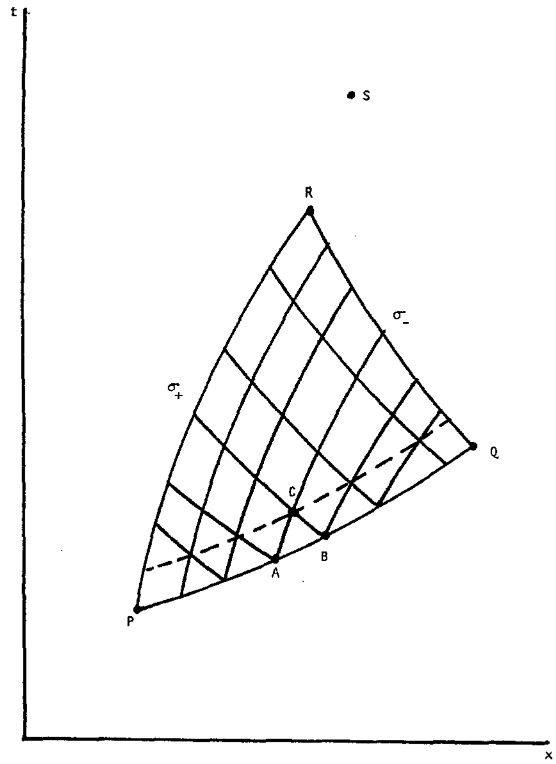

FIG. 1. The domain of dependence for data prescribed along a space-like curve PQ.

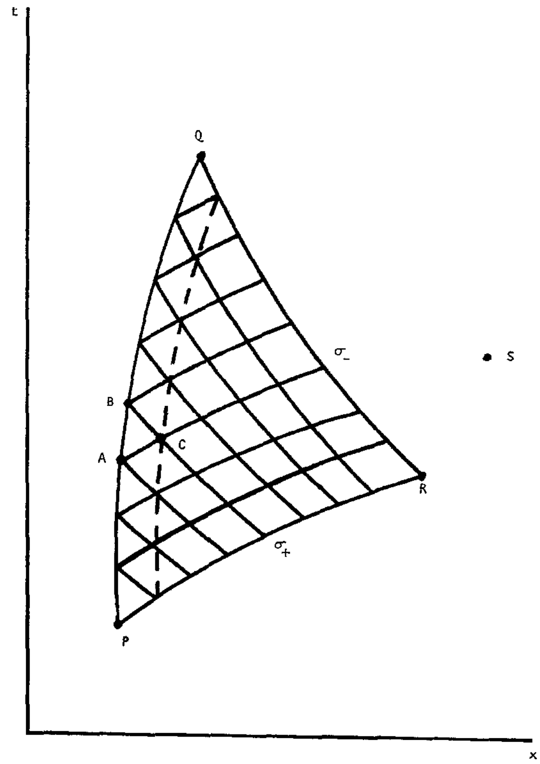

FIG. 2. The domain of dependence for data prescribed along a time-like curve PQ.

relations on the characteristics from A and B

$$v_C+L_C=v_A+L_A,\quad L_C-v_C=L_B-v_B,$$

which yield

$$v_C=\tfrac{1}{2}(L_A-L_B+v_A+v_B) \tag{38}$$
$$L_C=\tfrac{1}{2}(L_A+L_B+v_A-v_B).$$

Thus the data at A and B yield data at C. By choosing a sequence of pairs of points A, B sufficiently closely spaced on PQ we can build up the solution on a locus of C on one side of PQ, and by repeating the process we can construct the solution for ρ and v throughout the region PQR where PR is a plus-characteristic and QR a minus-characteristic. A similar process builds up the solution on the other side of PQ by interchanging the plus- and minus-characteristics. See Figs. 1 and 2. This method of solution is particularly well adapted to numerical integration of the equations of motion, and to programming on a digital computer.

The region PQR defined above is called the domain of dependence; to learn anything about the solution outside this domain requires additional initial information. Thus we cannot predict the flow at S because that point can be influenced both by signals from before P and after Q about which initial data are lacking.

MOTION OF FLUID WITH THERMAL CONDUCTIVITY

To illustrate more generalized applications of the method of characteristics, we now consider very briefly the problem of a compressible fluid with thermal conductivity but with no viscosity. The equations of motion must now be supplemented by the equation of conservation of heat:

$$\partial S/\partial t+v\partial S/\partial x=(K/T\rho)\partial^2T/\partial x^2, \tag{39}$$

where S is the entropy per unit mass, and T the absolute temperature. We now need two specifying equations; for example, to specify a gas we need both the equation of state and the heat capacity. Here we shall take S and ρ as the thermodynamic variables, and specify T and p as given functions of S and ρ: this means that the partial derivatives with respect to S and ρ, viz. T_S, T_ρ, p_S, and p_ρ are also given functions of ρ and S. Therefore we have

$$\partial p/\partial x = p_S \partial S/\partial x + p_\rho \partial\rho/\partial x, \tag{40}$$

to be used in Eq. (2) and

$$\begin{aligned}\partial^2 T/\partial x^2 = T_S \partial^2 S/\partial x^2 + T_\rho \partial^2\rho/\partial x^2 \\ + T_{SS}(\partial S/\partial x)^2 + 2T_{S\rho}(\partial S/\partial x)(\partial\rho/\partial x) \\ + T_{\rho\rho}(\partial\rho/\partial x)^2,\end{aligned} \tag{41}$$

to rewrite Eq. (39). Using a notation like that in Eq. (11) the three equations (1), (2), and (39) can be written in the form

$$\begin{aligned}\sum_k \mathbf{a}_{1k}\cdot\mathbf{D}u_k = 0, \\ \sum_k \mathbf{a}_{2k}\cdot\mathbf{D}u_k = 0, \\ \sum_k \mathbf{A}_k : \mathbf{DD}u_k = B,\end{aligned} \tag{42}$$

where $u_1 = \rho$, $u_2 = v$, $u_3 = S$ are the dependent variables, B is a term containing no second-order derivatives of the dependent variables, $\mathbf{a}_{ik}$ are the following vectors:

$$\begin{aligned}\mathbf{a}_{11} = (1,v); \quad \mathbf{a}_{12} = (0,\rho); \quad \mathbf{a}_{13} = 0; \\ \mathbf{a}_{21} = (0,p_\rho); \quad \mathbf{a}_{22} = (\rho,\rho v); \quad \mathbf{a}_{23} = (0,p_S),\end{aligned} \tag{43}$$

and $\mathbf{A}_k$ are the following dyadics:

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 \\ 0 & T_\rho \end{pmatrix}; \quad \mathbf{A}_2 = 0; \quad \mathbf{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & T_S \end{pmatrix}. \tag{44}$$

Because there are second-order derivatives involved in Eqs. (42), the boundary value problem is not as simple as before. We must now know not only the dependent variables along the "initial curve," but we must also know their first derivatives across the curve. We may again use the notation of Eq. (15), and write Eqs. (42) in the form

$$\left.\begin{aligned}\sum_k \mathbf{a}_{1k}\cdot\boldsymbol{\lambda}\partial u_k/\partial n = b_1, \quad \sum_k \mathbf{a}_{2k}\cdot\boldsymbol{\lambda}\partial u_k/\partial n = b_2, \\ \text{and} \qquad \sum_k \mathbf{A}_k : \boldsymbol{\lambda\lambda}\partial^2 u_k/\partial n^2 = B_3,\end{aligned}\right\}, \tag{45}$$

where b_1 and b_2 involve only derivatives in the $\boldsymbol{\tau}$ direction, and B_3 involves no second-order derivatives in the $\boldsymbol{\lambda}$ direction. The characteristic curves are now defined as those along which the initial data cannot be used to determine the second-order derivatives in the direction $\boldsymbol{\lambda}$ normal to the curve. To make the set of equations all second order, we operate on the first two with $\partial/\partial n$:

$$\begin{aligned}\sum_k \mathbf{a}_{1k}\cdot\boldsymbol{\lambda}\partial^2 u_k/\partial n^2 = B_1, \quad \sum_k \mathbf{a}_{2k}\cdot\boldsymbol{\lambda}\partial^2 u_k/\partial n^2 = B_2, \\ \sum_k \mathbf{A}_k : \boldsymbol{\lambda\lambda}\partial^2 u_k/\partial n^2 = B_3,\end{aligned} \tag{46}$$

where none of the B's contain any second-order derivative in the $\boldsymbol{\lambda}$ direction. We now have three simultaneous equations for the three second derivatives $\partial^2 u_k/\partial n^2$, when we are given the dependent variables and their first derivatives on some initial curve tangent to $\boldsymbol{\tau}$. They have *no* solution if $\boldsymbol{\lambda}$ is a root of the determinant equation

$$\begin{vmatrix} \mathbf{a}_{11}\cdot\boldsymbol{\lambda} & \mathbf{a}_{12}\cdot\boldsymbol{\lambda} & \mathbf{a}_{13}\cdot\boldsymbol{\lambda} \\ \mathbf{a}_{21}\cdot\boldsymbol{\lambda} & \mathbf{a}_{22}\cdot\boldsymbol{\lambda} & \mathbf{a}_{23}\cdot\boldsymbol{\lambda} \\ \mathbf{A}_1:\boldsymbol{\lambda\lambda} & \mathbf{A}_2:\boldsymbol{\lambda\lambda} & \mathbf{A}_3:\boldsymbol{\lambda\lambda} \end{vmatrix} = 0. \tag{47}$$

Using Eq. (43) and (44), this is simply

$$\begin{vmatrix} \lambda_0 + v\lambda_1 & \rho\lambda_1 & 0 \\ p_\rho\lambda_1 & \rho\lambda_0 + \rho v\lambda_1 & P_S\lambda_1 \\ T_\rho\lambda_1^2 & 0 & T_S\lambda_1^2 \end{vmatrix} = 0, \tag{48}$$

which yields for the normal to the characteristics:

$$\lambda_1^2 = 0, \quad \lambda_0/\lambda_1 = -v \pm (P_\rho - p_S T_\rho/T_S)^{\frac{1}{2}} = -v \pm c_T,$$

where c_T is the isothermal limit of the sound speed. The directional derivative along the characteristic is thus $\partial/\partial t + (v \pm c_T)\partial/\partial x = \partial/\partial\sigma_\pm$ and in each case the σ parameter is the signalling time with a propagating speed equal to the high-frequency limit (isothermal) sound speed. Since the characteristic corresponds to a discontinuous second derivative, it is physically natural that its propagation speed should be the high-frequency limit.

Actually to construct a solution to this flow problem, one must discuss in detail the compatibility relations

$$\sum_j \alpha B_j = 0, \tag{49}$$

where α_j are solutions of the equation

$$\alpha_1 \mathbf{a}_{1k}\cdot\boldsymbol{\lambda} + \alpha_2 \mathbf{a}_{2k}\cdot\boldsymbol{\lambda} + \alpha_3 \mathbf{A}_k : \boldsymbol{\lambda\lambda} = 0, \tag{50}$$

analogous to Eq. (24): viz.,

$$\alpha_1=\lambda_0/\lambda_1+v, \quad \alpha_2=-1, \quad \alpha_3=p_S/T_S\lambda_1. \quad (51)$$

There is one set for each characteristic direction. While the B's are somewhat cumbersome expressions, and it would not serve the purpose of this article to develop the algebraic details, it should now be clear that the flow problem can in principle be solved.

Here we would point out that there appears to be a basic distinction between the fluid without thermal conductivity, and the fluid with a small thermal conductivity that may approach zero. In the first idealized case the characteristics are lines in the $x-t$ plane across which the flow speed and the density may be discontinuous. But such characteristics do not exist in the thermal conducting fluid, where the characteristics are defined rather as lines across which discontinuities in only the first derivatives of the flow speed and density may exist. Another way of looking at this is to realize that the characteristics in the conducting fluid are obtained by the 3×3 determinant of Eq. (47), while for the nonconducting fluid they are found from the 2×2 determinant of Eq. (18). Allowing the conductivity to approach zero cannot convert a 3×3 gradually into a 2×2 determinant!

It is not even certain that the signal speeds on the characteristics become identical in the limit of zero conductivity; all we do know is that the frequency, beyond which the sound wave speed becomes isothermal, increases with decreasing conductivity. There is indeed a real qualitative difference between the characteristics of the very poor conductor and the truly nonconducting fluid.

This kind of difference is even more marked when we consider viscous fluids, and allow the viscosity η to approach zero. Viscosity adds a dissipation term to the heat equation, which simply modifies the term B on the right side of Eq. (42). The viscous term in the equation of momentum change is proportional to the second derivative of the speed, and Eq. (2) therefore becomes itself second order in the derivatives of the dependent variables. The terms appearing on the left of the second equation in Eq. (42) then are all transferred to the right side of the equation. The analog of Eq. (48) now turns out to be

$$\begin{vmatrix} \lambda_0+v\lambda_1 & \rho\lambda_1 & 0 \\ 0 & \eta\lambda_1^2 & 0 \\ T_\rho\lambda_1^2 & 0 & T_S\lambda_1^2 \end{vmatrix}=0,$$

and the characteristics are given by $\lambda_0+v\lambda_1=0$, i.e., $\tau_1/\tau_0=v$; or $\lambda_1=0$. The only paths in a viscous fluid across which the density gradient can be discontinuous are thus the particle histories; the signal speeds are the particle velocities. Even if the viscosity were to gradually approach zero, there is no way in which these particle paths could gradually approach the characteristics of the nonviscous fluid. These are examples of "asymptotic paradoxes"—well-known in hydrodynamics.[8]

The theory of characteristics in an ideal fluid neglecting both thermal conductivity and viscosity has been extensively applied to flows of real fluids, but the validity of the neglect has usually been assumed rather than critically examined. We conclude this paper with the unanswered question: How much meaning can be ascribed to the characteristics of an ideal nonconducting nonviscous flow, when applied to the flow of a real fluid with small viscosity and small thermal conductivity?

[8] G. Birkhoff, *Hydrodynamics* (Dover Publications, Inc., New York, 1950).