In the interval

$$G^2/2C < \mathcal{H} < G^2/2A,$$

the isoenergetic curves are no longer closed, and they no longer cross the axis $l$. With $\mathcal{H}$ decreasing towards $G^2/2C$, which is the value that the energy takes at the equilibrium $L=G$, the curves tend to flatten themselves onto the line $L=G$. Thus this equilibrium is also stable, and the body now "rotates" in the plane $Oxy$.

In the numerical example we chose $A=0.999378$, $B=0.999498$, and $C=G=1$. These corresponds to normalized values for the Moon. One also computes that

$$\mathcal{H}_A = 0.50031119,$$

$$\mathcal{H}_B = 0.50025112,$$

$$\mathcal{H}_C = 0.50000000.$$

In more intuitive terms, the isoenergetic curves look like the level curves of a typical landscape. The permanent rotation around the axis $Ox$ of smaller inertia appears at the tope of a hill; the permanent rotation around the axis $Oy$ of middle inertia constitutes a pass half way on the slope of the hill. The northern and southern sides of the landscape are made of two longitudinal valleys at the bottom of which flows the singular line representing the permanent rotation around the axis $Oz$ of larger inertia.

Such a phase representation of the Euler–Poinsot problem bears striking analogy with that of a simple pendulum.

**ACKNOWLEDGMENT**

The computations leading to Fig. 2 were implemented by C. Frederick Peters, graduate student at the Department of Astronomy, Yale University.

# Normal Shock Waves in a Compressible Fluid*

Ronald Blum

*The College and Department of Physics, The University of Chicago, Chicago, Illinois 60637*

(Received 2 August 1966; revision received 22 December 1966)

The fundamental principles and relations governing normal shock waves in a compressible fluid, in both the classical and hydromagnetic cases, are stated and derived in a manner suitable for inclusion in an undergraduate physics curriculum. Acoustic and magnetoacoustic velocities are derived as limiting cases of very weak shocks, and the second law of thermodynamics is employed to demonstrate that fluid flow through a stationary shock wave always requires a supersonic-to-subsonic transition.

## INTRODUCTION

In a recent review[1] of texts in fluid dynamics, the reviewer noted: "It is unfortunate that physicists have generally abandoned the physics of fluids as a research frontier to the mathematician, who regards a fluid as a nonphysical continuum, and to engineers, who stress the empirical means of solving practical problems." Another commentator[2] points out that "A paradox confronts contemporary physicists; although scope, importance, and utility of fluid dynamics in physics have grown spectacularly, teaching of the subject in physics departments has been receiving less and less attention." In particular, it is often the case that the subject of shock waves in compressible fluids—so vitally important in the present era of space physics, plasma physics, and high-speed flows—is almost[3] completely neglected in the undergraduate physics curriculum and rarely mentioned in the graduate curriculum. Furthermore, the customary discussion of shock waves, with its emphasis on Mach numbers (which look like constants to the uninitiated) is unsympathetic to the physicist who prefers to

* Work performed under National Aeronautics and Space Administration Grant No. NASA 179-61.

[1] R. J. Seeger, Am. J. Phys. **34**, 173 (1966).

[2] R. G. Fowler, Phys. Today **19**(6), 37 (1966).

[3] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley Pub. Co., Inc., Reading, Mass., 1963), Vol. 1, Chap 51, is a notable exception.

see explicit dependence on physical quantities. Hence this paper attempts to present the shock wave, both classical and hydromagnetic, in its simplest form, appropriate for inclusion in an undergraduate physics curriculum. The transport equations of fluid dynamics are used in an integrated form, and the use of Mach numbers, except for the sake of completeness, has purposely been eschewed. (The customary treatment in terms of Mach numbers may be found in any good text on the subject.[4-6])

A shock wave can be understood by analogy with a breaking water wave: since the velocity of a water wave is proportional to the depth of the water below it, significant differences appear in the propagation velocities at different heights in the wave once the crest height becomes comparable to the depth; the crest steepens and finally breaks. In the case of a shock wave in a compressible fluid, a disturbance causes a pressure wave to propagate. However, in a finite disturbance, as distinct from an ideal sound wave, the pressure wave heats the medium as it passes so that the acoustic velocity increases over the duration of the disturbance. Imagine the disturbance as a succession of infinitesimal pulses, each one producing an infinitesimal wavelet which, passing through the medium already elevated in temperature due to preceding wavelets, propagates more rapidly than its predecessors and overtakes them. Imagine, also, a convective effect due to the fact that a wavelet propagates at the local sound velocity relative to the medium, already set in motion by the preceding disturbances. Thus, analogous to the steepening of the water wave, there is a compressional wave forming whenever the signal speed of a wavelet is increased by the effect of preceding disturbances. However, in the case of the gas, compression (steepening) ceases when the wave reaches a critical thickness (of the order of 10 mean free paths), so that the large temperature and velocity gradients within the wave set limits to its thickness and eventually cause it to decay as energy is dissipated through viscous losses and heat conduction.

To analyze this situation the shock wave is considered as a thin region—a surface of discontinuity for mathematical purposes—on either side of which the gas is taken to be in thermal equilibrium on the assumption that the duration of equilibrium on either side of the shock is long as compared with the transit time of the shock The one-dimensional "jump conditions" across a normal shock wave (one in which the wave front is perpendicular to the direction of propagation) are mathematically simple and can be included quite naturally in a classical junior or senior-level course in Thermodynamics, following the discussion of the Joule–Thomson effect, under the rubric "Steady-State Flows." This problem also serves to derive the velocity of acoustic waves as the limiting case of a very weak shock wave, and to illustrate the utility of the Second Law of Thermodynamics in determining the direction of an irreversible process. With the inclusion of magnetic terms the theory can be applied to hydromagnetic shocks and to derive the magnetoacoustic wave velocity.

The theoretical importance of the shock wave lies in the fact that real waves are finite in amplitude; their practical importance is well known from the phenomena of blast waves, supersonic flow, sonic booms, and, more recently, their application to the motion of satellites and other bodies moving through the interplanetary and interstellar media. As a research tool, shock waves furnish a significant, if transient, means of rapidly and homogeneously heating gases to temperatures high enough to cause dissociation and ionization. It is hoped that even the brief treatment given here may broaden the student's grasp of nonequilibrium nonlinear processes and lessen the possibility that this important and timely area of physics be neglected by default.

## I. STEADY-STATE TRANSPORT EQUATIONS

As in the case of the Joule–Thomson effect, the flow of fluid through a region (see Fig. 1) containing the shock, on whose boundaries thermodynamic equilibrium prevails, is considered. The problem is analyzed in the shock system in which the wave is at rest, and the channel is

---

[4] H. W. Liepmann, and A. Roshko, *Elements of Gasdynamics* (John Wiley & Sons, Inc., New York, 1957).
[5] J. N. Bradley, *Shock Waves in Chemistry and Physics* (John Wiley & Sons, Inc., New York, 1962).
[6] J. W. Bond, Jr., K. M. Watson, and J. A. Welch, Jr., *Atomic Theory of Gas Dynamics* (Addison–Wesley Publ. Co., Inc., Reading, Mass., 1965).

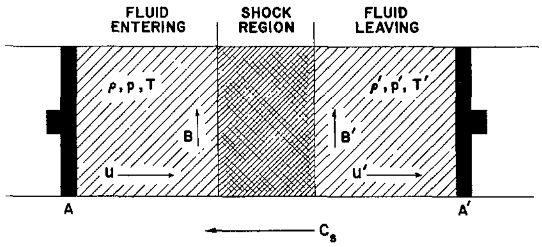


FIG. 1. Normal one-dimensional shock wave (shock region) shown in the shock system. $T$, temperature; $p$, pressure; $\rho$, density; $u$, fluid velocity; $B$, magnetic-field intensity. Direction of propagation of shock in medium initially at rest indicated by arrow $c_s$.

assumed to be of unit area; conservation of mass requires that the same amount of matter as enters leave the shock region per second:

$$\rho u = \rho' u' = J_0, \tag{1}$$

where $\rho$ is mass density, $u$ is stream velocity, $J_0$ is (constant) mass flux density, and primed values refer to quantities downstream of the shock.

Consider an element of fluid, mass $\rho u$, which crosses the left boundary of the shock region in one second: by the time it emerges at the right boundary of the shock region, its momentum has changed due to the forces acting upon it in the shock region (the Second Law is used to prove that they are retarding forces). Since the fluid is not accelerating at the boundaries of the shock region, the hydrostatic pressures must exactly balance the net effect of the complex of forces acting within the shock region, which cause the momentum of the fluid element to change from $\rho u^2$ to $\rho' u'^2$. Hence,

$$\begin{pmatrix}\text{net rate of change of}\\ \text{momentum per second}\end{pmatrix} = \rho' u'^2 - \rho u^2$$

$$= p - p' = \begin{pmatrix}\text{net accelerating}\\ \text{force on element}\end{pmatrix}.$$

(As an aid to intuition, it is noted in advance that both sides of this equation will be negative.) This equation may be rewritten

$$\rho u^2 + p = \rho' u'^2 + p' = \Phi_0, \tag{2}$$

where $p$ is gas pressure and $\Phi_0$ is a constant.

The third transport equation is obtained from the conservation of energy, and is derived as in the Joule–Thomson effect, but with the addition of a term, $\frac{1}{2}(\rho u)u^2$, to represent the transport of kinetic energy by the fluid. Imagine pistons located at $A$ and $A'$ forcing the fluid through the shock, and set

$$\begin{pmatrix}\text{net change in internal plus}\\ \text{kinetic energy of element } \rho u\end{pmatrix}$$

$$= \begin{pmatrix}\text{net work done by}\\ \text{pistons on } \rho u\end{pmatrix};$$

thus:

$$\rho' u'(E'/M + \tfrac{1}{2}u'^2) - \rho u(E/M + \tfrac{1}{2}u^2) = pu - p'u',$$

where $E$ is internal energy/mole and $M$ is molecular weight. This gives the equation of energy transport after substituting from (1):

$$E + pM/\rho + \tfrac{1}{2}Mu^2$$
$$= E' + p'M/\rho' + \tfrac{1}{2}Mu'^2 = H_0; \tag{3}$$

where it is recalled that enthalpy,

$$H = E + pM/\rho = C_p T,$$

$C_p$ is molar specific heat at constant pressure, and $H_0$ is (constant) stagnation enthalpy/mole.

Equations (1), (2), and (3) contain four variables, $u$, $\rho$, $p$, and, for a perfect gas, $T$. If we assume that the conditions on one side of the shock are generally given, then one more equation is needed to solve the system and find the conditions on the other side—the equation of state:

$$p = (\rho/M)RT, \tag{4}$$

where $R = C_p - C_v$ is the universal gas constant, and $C_v$ the molar specific heat at constant volume. The system is now complete and may be solved.

## II. SOLUTION OF TRANSPORT EQUATIONS

Given velocity, temperature, and pressure on either side of the shock, the system of equations can be solved in terms of the constants $J_0$, $\Phi_0$, $H_0$ evaluated from the given conditions. Thus, a quadratic equation for $u$ is obtained by substituting Eqs. (1), (2), (4), and $H = C_p T$ into Eq. (3):

$$u^2(C_p/R - \tfrac{1}{2}) - uC_p\Phi_0/RJ_0 + H_0/M = 0. \tag{5}$$

This equation has two valid solutions,

$$u, u' = \frac{C_p\Phi_0/J_0}{C_p + C_v}[1 \pm \Delta]; \tag{6}$$

where

$$\Delta^2 = 1 - (4H_0R^2J_0^2/MC_p^2\Phi_0^2)(C_p/R - \tfrac{1}{2}), \quad (7)$$

and since $C_p/R > \frac{1}{2}$, all the constant coefficients in Eq. (5) are positive; hence, by Descartes' sign rule,[7] both roots are real and positive, $0 \leq \Delta^2 < 1$. Furthermore, if one substitutes for $J_0$, $\Phi_0$, $H_0$ in terms of physical quantities known on one side of the shock region, one finds that $\Delta = 0$ only in the special case

$$u'^2 = u^2 = \gamma RT/M = c^2, \quad (8)$$

where $c$ is the speed of sound, and $\gamma = C_p/C_v$. Thus, an ideal sound wave may be viewed as the limiting case of a very weak shock, and substitution in the equations of transport shows that the other physical parameters are undisturbed by the passage of the shock. (Note: shock *strength* is usually defined by the ratio $p'/p$ or $p'/p - 1$.)

Even if given only two pieces of information, concerning temperature and velocity alone, it is still possible to eliminate pressure and density from Eqs. (2) and (3) to obtain

$$u + RT/Mu = u' + RT'/Mu' = \Phi_0/J_0, \quad (9)$$

$$C_pT + \tfrac{1}{2}Mu^2 = C_pT' + \tfrac{1}{2}Mu'^2 = H_0; \quad (10)$$

which may, depending on the circumstances, be solved for $T(u,u')$ and $T'(u,u')$:

$$T = (Mu/RC_p)[C_pu' - R(u'+u)/2], \quad (11)$$

$$T' = (Mu'RC_p)[C_pu - R(u+u')/2]; \quad (12)$$

or for $u'(T,u)$, $T'(T,u)$:

$$u' = uR(2C_pT/Mu^2+1)/(C_p+C_v), \quad (13)$$

$$T' = T + (M/2C_p)(u^2 - u'^2); \quad (14)$$

and, without any additional information, one may calculate the density ratio and the shock strength

$$\frac{\rho'}{\rho} = \frac{u}{u'}; \quad \frac{p'}{p} = \left(\frac{Mu^2}{RT}\right)\left(1 - \frac{u'}{u}\right) + 1. \quad (15)$$

The equations expressing the change in physical parameters crossing the shock are known as Rankine–Hugoniot equations; since the original equations are symmetric with respect to the shock, primed variables may be substituted for unprimed variables in the above formulas, and vice versa.

The conservation equations are satisfied whether $u < u'$ or $u > u'$, although intuition expects the latter. To see that $u > u'$, note that passage through the shock region is an irreversible process and must result in an increase of entropy in the gas (which is an isolated thermodynamic system). If the molar entropy, $S$, of an ideal gas is expressed in terms of its enthalpy it is found

$$S' - S = C_p \ln(H'/H) - R \ln(p'/p) \geq 0. \quad (16)$$

From Eqs. (2) and (3),

$$\begin{aligned} H'/H &= 1 + (M/2H)(u^2 - u'^2); \\ &= 1 + M(u^2 - u'^2)/2C_pT \\ p'/p &= 1 + (\rho u^2 - \rho' u'^2)/p \\ &= 1 + Mu(u - u')/RT \end{aligned} \quad (17)$$

is obtained and Eq. (11) may be used for $T$ in Eq. (17). If the above expressions are substituted into Eq. (16), the mathematical behavior of the entropy change $(S'-S)/R = f(x)$ can be investigated as a function of $x = u'/u$:

$$(\gamma - 1)f(x) = \gamma \ln x + \ln[\gamma + 1 - (\gamma - 1)x] - \ln[(\gamma + 1)x + 1 - \gamma]. \quad (18)$$

From this equation it is immediately obvious that in the case of a sound wave $u' = u = c$, $x = 1$, there is no entropy change, $f(1) = 0$.

To locate the extrema of $f(x)$, set

$$\frac{df}{dx} = \frac{-\gamma(\gamma+1)(x-1)^2}{x[\gamma+1-(\gamma-1)x][(\gamma+1)x+1-\gamma]} = 0. \quad (19)$$

Thus, $f'(1) = 0$ is the only extremum, and since $f'(x) \leq 0$ for

$$(\gamma - 1)/(\gamma + 1) < x < (\gamma + 1)/(\gamma - 1)$$

then $f(x)$ is monotonic, decreasing in this range, and the only allowed values of $x$ are those for which $0 \leq f(x) < \infty$; i.e.,

$$(\gamma - 1)/(\gamma + 1) < x \leq 1. \quad (20)$$

Hence, $u' \leq u$, as expected, the lower limit being approached in the limiting case of an infinitely strong shock, $u \gg c$.

It is clear from the second-order zero in Eq. (19) that the second derivative also vanishes for

---

[7] J. R. Britton and L. C. Snively, *Algebra* (Rinehart and Co., Inc., New York, 1950), p. 385.

$x=1$. Thus, a series expansion of $f(x)$ about this point yields

$$S'-S \doteq (R/12)\gamma(\gamma+1)(1-u'/u)^3 \qquad (21)$$

so that for weak shocks, $u \doteq c$, the change of state is nearly isentropic.

The shock system is the natural one to use in many situations where so-called "standing shocks" are produced, the source of the disturbance being considered stationary—e.g., wind-tunnel models, re-entry bodies, laboratory plasmas, and the "bow wave" standing between the earth's magnetic field and the expanding solar corona. For other applications it may be advantageous to consider the shock wave as propagating through the initially stationary medium—e.g., blast waves and shock tubes. To do this the shock wave is considered as propagating with some velocity $c_s$, and the fluid behind it is imagined to be pushed along by a piston with velocity $u_p$. The Rankine–Hugoniot equations are then transformed by substituting

$$u=c_s,\ u'=u-u_p=c_s-u_p. \qquad (22)$$

It is customary to express the Rankine–Hugoniot equations in terms of Mach number, defined as the ratio of local fluid velocity to local sound velocity: $M_1=u/c$; $M_2=u'/c'$. This yields

$$\rho/\rho'=u'/u=[2+(\gamma-1)M_1^2]/(\gamma+1)M_1^2; \qquad (23)$$

$$T'/T=1+[2(\gamma-1)/(\gamma+1)^2] \times[(\gamma M_1^2+1)/M_1^2](M_1^2-1); \qquad (24)$$

$$p'/p=1+[2\gamma/(\gamma+1)](M_1^2-1). \qquad (25)$$

These equations show that the critical parameter in shock-wave phenomena is the ratio of the kinetic energy of directed motion to the thermal kinetic energy, since $M_1^2 \propto Mu^2/RT$. From Eq. (23) it is noted that the allowed values of $u'/u$ correspond to $1 \leq M_1^2 < \infty$ and $(\gamma-1)/2\gamma < M_2^2 \leq 1$. This means that in the shock system the transition must always be from supersonic to subsonic (i.e., with reference to the *local* sound velocity).

### III. HYDROMAGNETIC SHOCK WAVE, TRANSVERSE MAGNETIC FIELD

Consider the case of a fluid with infinite electrical conductivity; if the fluid is a neutral plasma consisting of ions and free electrons the following equations apply to the ions which constitute the bulk of the fluid. Since any motion of the fluid across magnetic field lines would give rise to infinite currents due to induced emf's, infinite conductivity requires that the magnetic field is embedded, "frozen," in the material, and constrained to move along with it. Hence the flux density $B$ must vary directly with the density of the fluid:

$$B'/B=\rho'/\rho; \qquad (26)$$

which, since Eq. (1) is unchanged, gives rise to the "induction" condition across the shock

$$Bu=B'u'=\mathcal{E}_0, \qquad (27)$$

where $\mathcal{E}_0=$const, with dimensions of electric-field intensity. Equation (27) can also be derived in an equivalent way by considering that an observer in the shock system feels an electric field $\mathcal{E}_0=|-\mathbf{u}\times\mathbf{B}|=uB$ due to the transverse magnetic field flowing past him with the fluid velocity. This magnetic field is tangential to the shock (considered as a mathematical surface), therefore it must be the same on both sides of it, hence Eq. (27).

Since the magnetic field participates in the flow it must also be included in the energy and momentum equations. While the concept of magnetic energy density $B^2/8\pi$ (cgs units) is generally understood after the student's first or second year of physics,[8] that of magnetic pressure is somewhat more difficult. In the case of frozen-flow, one can picture the flow and the field transverse to it as moving along like a troop of lancers; the total flux (lances) remains constant, but the flux density changes as the flow contracts or expands (as the troop closes up or spreads out). If a fluid element of dimensions $x$, $y$, $z$ ($u\|x, B\|z$) is compressed by $dx$, the constancy of the flux implies $d(Bx)=0$ which in turn implies an increase in the magnetic energy of the volume element:

$$dE_m=yzd(B^2x/8\pi)=-yz(B^2/8\pi)dx=-yzp_xdx,$$

where $p_x=B^2/8\pi$ is the pressure or force/area with which the magnetic field resists compression in the $x$ direction.

Although Eqs. (1) and (4) are unchanged, Eqs. (2) and (3) must be altered by the addition

[8] E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill Book Co., New York, 1965), Chap. 7.

of terms equal to $B^2/8\pi$ to *both* $p$ and $\rho E/M$ to account for the pressure and energy density of the magnetic field. This gives:

$$\rho u^2+p+B^2/8\pi=\rho' u'^2+p'+B'^2/8\pi=\Phi_m, \quad (28)$$

$$E+Mp/\rho+\tfrac{1}{2}Mu^2+MB^2/4\pi\rho = E'+Mp'/\rho'+\tfrac{1}{2}Mu'^2 + MB'^2/4\pi\rho'=H_m, \quad (29)$$

where $\Phi_m$, $H_m$ are constants.

The set of five equations (1), (4), (26), (28), and (29) can be reduced, by substitution, to a cubic in $u$:

$$u^3(C_p-R/2)-u^2\Phi_m C_p/J_0+uRH_m/M + (\mathcal{E}_0^2/8\pi J_0)(C_p-2R)=0, \quad (30)$$

and if $\mathcal{E}_0$ vanishes the quadratic of Eq. (5) is recovered. Since all of the constant coefficients are positive, Descartes' rule of signs shows that that there are one negative and two positive roots of Eq. (30). The negative root may be discarded as unphysical, and if the fluid velocity (call it $U$) is known on one side of the shock, then $u=U$ is one root of Eq. (30). Thus, this equation can be reduced to a quadratic by dividing by $(u-U)$:

$$(C_p-R/2)J_0u^2+[(C_p-R/2)J_0U-\Phi_mC_p]u + (C_p-R/2)J_0U^2-\Phi_mC_pU + RJ_0H_m/M=0, \quad (31)$$

an equation easily solved once the coefficients have been evaluated from initial conditions.

There is an important special case for a monatomic perfect gas of charged particles of mass $m$, charge $q$: if the cyclotron frequency $qB/m$ of the ions is much greater than their collision frequency, they will spiral around field lines which inhibit motion radially outward. Thus, the ions have only two degrees of freedom and $C_p=2R$. In this case the last term of Eq. (30) vanishes; what remains is a quadratic:

$$3u^2-4u\Phi_m/J_0+2H_m/M=0, \quad (32)$$

which has two positive roots.

It has been shown, for a classical shock, that an infinitesimally weak shock corresponds to an acoustic wave. The case of a magnetoacoustic wave is now examined by substituting $u'-u=du$, $p'-p=dp$, etc.

The appropriate equations become

$$d(\rho u)=\rho du+ud\rho=0; \quad (33)$$

$$d(Bu)=Bdu+udB=0; \quad (34)$$

$$d(\rho u^2)+dp+BdB/4\pi=0; \quad (35)$$

$$dH+Mudu+(M/4\pi)d(B^2/\rho)=0. \quad (36)$$

Equations (33), (34), and (35) may be solved for $d\rho$, $dB$, and $dp$:

$$d\rho=-(\rho/u)du, \quad dB=-(B/u)du,$$
$$dp=(B^2/4\pi u)du-\rho udu,$$

and the results substituted into Eq. (36) in order to show that the differential of entropy still vanishes:

$$TdS=dH-Mdp/\rho=0, \quad (37)$$

and the hydromagnetic shock wave is isentropic, at least to first order. If Eqs. (33) and (34) are substituted into (35), it is found that, after replacing $du$ by $dp$ and $d\rho$,

$$(B^2/4\pi\rho-u^2)d\rho+dp=0. \quad (38)$$

Thus, the magnetoacoustic velocity, which corresponds to the speed of sound in the classical case, is given by

$$u^2=c_{mu}{}^2=\left(\frac{dp}{d\rho}\right)_S+\frac{B^2}{4\pi\rho}=\frac{\gamma RT}{M}+\frac{B^2}{4\pi\rho}. \quad (39)$$

In the limiting case $B\rightarrow 0$, $u\rightarrow c=(dp/d\rho)_S$, the speed of sound.

For a very tenuous plasma such as the interplanetary gas or for very strong magnetic fields

$$u\rightarrow c_a=B/(4\pi\rho)^{\frac{1}{2}},$$

the Alfvén velocity, characteristic of waves when particles of the fluid interact chiefly through the medium of the magnetic field, the effect of collisions being negligible. This is the case of the "collisionless" shock in which the collective behavior of the particles as a fluid is due to the magnetic field's gluing them together. An excellent example of this phenomenon is afforded by the solar wind which forms a shock wave in flowing around the earth's magnetic field. This is possible, although the classical mean free path of ions in the solar wind is several times greater than the earth–sun distance, because the gyroradius of particles in the field is of the order of 500 km, which is much less than the dimensions of the shock wave, which is semicircular in form, with a radius of the order of 100 000 km, about 15 earth radii.