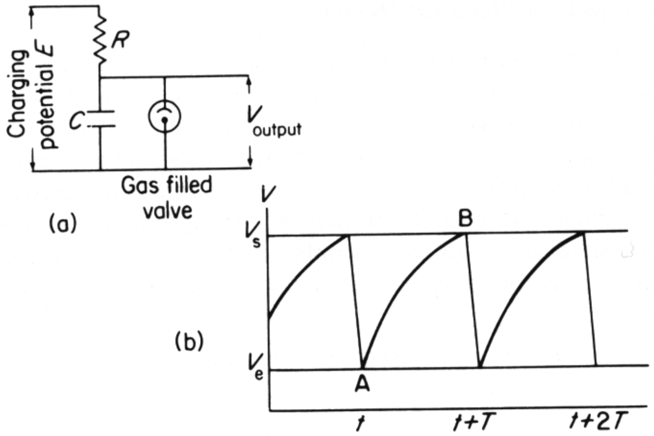

Fig. 11.6. Electrical circuit of a non-linear 'relaxation oscillator'. A capacitance C is charged through a resistance R to a potential $V_s < E$, at which the gas-filled valve strikes and rapidly discharges the condenser to an extinction potential V_e, when the valve ceases to conduct and the cycle is repeated

The capacitance charges to the potential V_s in a time τ so that

$$V_s = E - i_0 R\,\mathrm{e}^{-(t+\tau)/RC}$$

giving

$$\begin{aligned} V_s - V_e &= i_0 R(\mathrm{e}^{-t/RC} - \mathrm{e}^{-(t+\tau)/RC}) \\ &= i_0 R\,\mathrm{e}^{-t/RC}[1 - \mathrm{e}^{-\tau/RC}] \\ &= (E - V_e)[1 - \mathrm{e}^{-\tau/RC}] \end{aligned}$$

giving

$$\mathrm{e}^{-\tau/RC} = \frac{E - V_s}{E - V_e}$$

or

$$\tau = RC\left[\log_e\left(\frac{E - V_e}{E - V_s}\right)\right]$$

The period of oscillation is therefore directly proportional to the charging time constant RC.

A more sophisticated circuit produces a linear charging system with a very short discharge time so that the exponential voltage output becomes linear and gives a 'sawtooth' waveform. From Chapter 9 we know that this periodic function contains many harmonics. A sawtooth voltage output applied to the time base of an oscilloscope produces a linear sweep of the spot across the tube.

Non-linear Effects in Acoustic Waves

The linearity of the longitudinal acoustic waves discussed in Chapter 5 required the assumption of a constant bulk modulus

$$B = -\frac{\mathrm{d}P}{\mathrm{d}V/V}.$$

If the amplitude of the sound wave is too large this assumption is no longer valid and the wave propagation assumes a new form. A given mass of gas undergoing an adiabatic change obeys the relation

$$\frac{P}{P_0} = \left(\frac{V_0}{V}\right)^\gamma = \left[\frac{V_0}{V_0(1+\delta)}\right]^\gamma$$

in the notation of Chapter 5, so that

$$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x} = -\gamma P_0(1+\delta)^{-(\gamma+1)}\frac{\partial^2 \eta}{\partial x^2}$$

since $\delta = \partial\eta/\partial x$.

Since $(1+\delta)(1+s) = 1$, we may write

$$\frac{\partial p}{\partial x} = -\gamma P_0(1+s)^{\gamma+1}\frac{\partial^2 \eta}{\partial x^2}$$

and from Newton's second law we have

$$\frac{\partial p}{\partial x} = -\rho_0\frac{\partial^2 \eta}{\partial t^2}$$

so that

$$\frac{\partial^2 \eta}{\partial t^2} = c_0^2(1+s)^{\gamma+1}\frac{\partial^2 \eta}{\partial x^2}, \quad \text{where } c_0^2 = \frac{\gamma P_0}{\rho_0}$$

Physically this implies that the local velocity of sound, $c_0(1+s)^{(\gamma+1)/2}$, depends upon the condensation s, so that in a finite amplitude sound wave regions of higher density and pressure will have a greater sound velocity, and local disturbances in these parts of the wave will overtake those where the values of density pressure and temperature are lower.

A single sine wave of high amplitude can be formed by a close fitting piston in a tube which is pushed forward rapidly and then returned to its original position. Fig. 11.7a shows the original shape of such a wave and 11.7b shows the distortion which follows as it propagates down the tube. If the distortion continued the wave form would eventually appear as in fig. 11.7c, where analytical solutions for pressure, density and temperature would be multivalued, as in the case of the non-linear oscillator of fig. 11.3c. Before this situation is reached, however, the wave form stabilizes into that of fig. 11.7d,

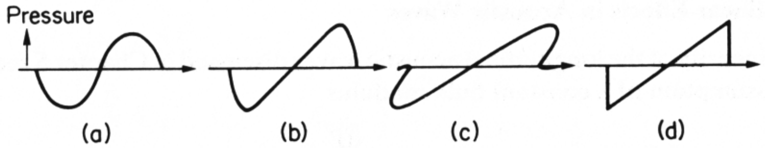

Fig. 11.7. The local sound velocity in a high amplitude acoustic wave (a) is pressure and density dependent. The wave distorts with time (b) as the crest overtakes the lower density regions. The extreme situation of (c) is prevented by entropy-producing mechanisms and the wave stabilises to an N type shock-wave (d) with a sharp leading edge

where at the vertical 'shock front' the rapid changes of particle density, velocity and temperature produce the dissipating processes of diffusion, viscosity and thermal conductivity. The velocity of this 'shock front' is always greater than the velocity of sound in the gas into which it is moving, and across the 'shock front' there is always an increase in entropy. The competing effects of dissipation and non-linearity produce a stable front as long as the wave retains sufficient energy. The N-type wave of fig. 11.7d occurs naturally in explosions (in spherical dimensions) where a blast is often followed by a rarefaction.

The growth of a shock front may also be seen as an extension of the Doppler effect (page 135), where the velocity of the moving source is now greater than that of the signal. In fig. 11.8a as an aircraft moves from S to S′ in a time t the air around it is displaced and the disturbance moves away with the local velocity of sound v_S. The circles show the positions at time t of the sound wave fronts generated at various points along the path of the aircraft but if the speed of the aircraft u is greater than the velocity of sound v_S regions of high density and pressure will develop, notably at the edges of the aircraft structure and along the conical surface tangent to the successive wave fronts which are generated at a speed greater than sound and which build up to a high amplitude to form a shock. The cone, whose axis is the aircraft path, has a half angle α where

$$\sin\alpha = \frac{v_S}{u}$$

It is known as the 'Mach Cone' and when it reaches the ground a 'supersonic bang' is heard.

The growth of the shock at the surface of the cone may be seen by considering the sound waves in fig. 11.8b generated at points A (time t_A) and B (time t_B) along the path of the aircraft, which travels the distance $AB = x = u\Delta t$ in the time interval $\Delta t = t_B - t_A$. The sound waves from A will travel the distance r_0 to reach the point P at a time

$$t_0 = t_A + \frac{r_0}{v_S}$$

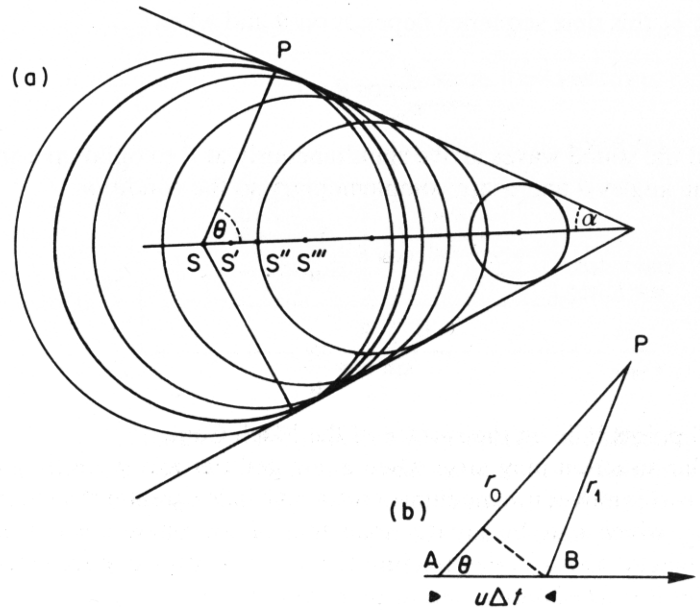

Fig. 11.8. (a) The circles are the wave fronts generated at points S along the path of the aircraft, velocity $u > v_s$ the velocity of sound. Wave fronts superpose on the surface of the Mach Cone (typical point P) of half angle $\alpha = \sin^{-1} v_s/u$ to form a shock front. (b) At point P sound waves arrive simultaneously from positions A and B along the aircraft path when $(u/v_s)\cos\theta = 1$. $(\theta + \alpha = 90°)$

Those from B will travel the distance r_1 to P to arrive at a time

$$t_1 = t_B + \frac{r_1}{v_S}$$

If x is small relative to r_0 and r_1, we see that

$$r_1 - r_0 \approx x\cos\theta = u\Delta t\cos\theta$$

so the time interval

$$t_1 - t_0 = t_B - t_A + \frac{(r_1 - r_0)}{v_S}$$

$$= \Delta t - \frac{u\Delta t\cos\theta}{v_S} = \Delta t\left(1 - \frac{u\cos\theta}{v_S}\right)$$

For the aircraft speed $u < v_S$, $t_1 - t_0$ is always positive and the sound waves arrive at P in the order in which they were generated.

For $u > v_S$ this time sequence depends on θ and when

$$\frac{u}{v_S}\cos\theta = 1$$

$t_1 = t_0$ and the sound waves arrive simultaneously at P to build up a shock.

Now the angles θ and α are complementary so the condition

$$\cos\theta = \frac{v_S}{u}$$

defines

$$\sin\alpha = \frac{v_S}{u}$$

so that all points P lie on the surface of the Mach Cone.

A similar situation may arise when a charged particle q emitting electromagnetic waves moves in a medium of refractive index greater than unity with a velocity v_q which may be greater than that of the phase velocity v of the electromagnetic waves in the medium ($v < c$). A Mach Cone for electromagnetic waves is formed with a half angle α where

$$\sin\alpha = \frac{v}{v_q}$$

and the resulting 'shock wave' is called Cerenkov radiation. Measuring the effective direction of propagation of the Cerenkov radiation is one way of finding the velocity of the charged particle.

Shock Front Thickness

The extent of the region over which the gas properties change, the shock front thickness, may be only a few mean free paths in a monatomic gas because only a few collisions between atoms are necessary to exchange the energy required to raise them from the equilibrium conditions ahead of the shock to those behind it. In a polyatomic gas the collisions are effective in producing a rapid increase in translational and rotational mode energies, but vibrational modes take much longer to reach their new equilibrium, so that the shock front thickness is very much greater.

Within the shock front thickness the state of the gas is not easily found, but the state of the gas on one side of the shock may be calculated from the state of the gas on the other side by means of the conservation equations of mass, momentum and energy.

Equations of Conservation

In a laboratory, shock waves are produced in a tube which is divided by a diaphragm into a short high-pressure section and a much longer low-pressure section. When the diaphragm bursts the expanding high pressure gas behaves

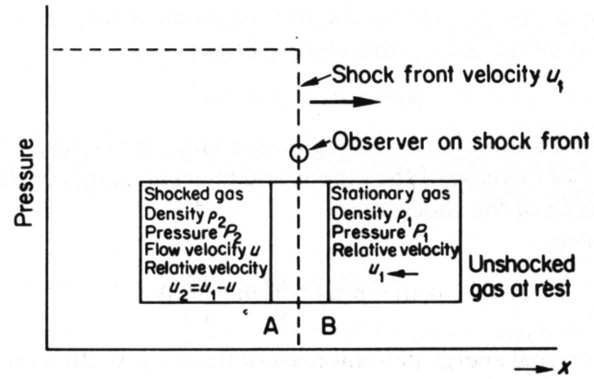

Fig. 11.9. The pressure 'step profile' of a shock wave developed in a shock tube is shown by the dotted line. The plane cross-sections at A and B remain fixed with respect to the observer O moving with the shock front at velocity u_1 into unshocked gas at rest of pressure p_1 and density ρ_1. The shocked gas has a pressure p_2, a density ρ_2 and a velocity u, with a relative velocity $u_2 = u_1 - u$ with respect to the shock front. The states of the gas at A and B are related by the conservation equations of mass, momentum and energy across the shock front. Experimental measurement of the shock velocity u_1 is sufficient to determine the unknown parameters if the stationary gas parameters are known

as a very fast low-interia piston which compresses the low pressure gas on the other side of the diaphragm and drives a shock wave down the tube. The profile of this shock wave is the step function shown as the dotted line in fig. 11.9, and the gas into which the shock is propagating is considered to be at rest. This simplifies the analysis, for we can consider the situation in fig. 11.9 as it appears to an observer O travelling with the shock front velocity u_1 into the stationary gas. The shock front is located within the region bounded by the surfaces A and B of unit area, each of which remains fixed with respect to the observer. The stationary gas which moves through the shock front from surface B acquires a flow velocity $u < u_1$ and a velocity relative to the shock front of $u_2 = u_1 - u$. From the observer's viewpoint the quantity of gas flowing into unit area of the region AB per unit time is $\rho_1 u_1$, where ρ_1 is the density of the gas ahead of the shock. The quantity leaving unit area of AB per unit time is $\rho_2(u_1 - u) = \rho_2 u_2$, where ρ_2 is the density of the shocked gas.

Conservation of mass yields $\rho_1 u_1 = \rho_2 u_2 = m$ (a constant mass). The force per unit area acting across the region AB is $p_2 - p_1$, which equals the rate of change of momentum of the gas within the unit element, which is $m(u_1-u_2)$. The conservation of momentum is therefore given by

$$p_1+\rho_1 u_1^2=p_2+\rho_2 u_2^2.$$

The work done on unit area of the region per unit time is $p_1u_1-p_2u_2$, and this equals the rate of increase of the kinetic and internal energy of the gas passing through unit area of the shock wave.

The difference

$$p_1u_1-p_2u_2=\frac{p_1}{\rho_1}m-\frac{p_2}{\rho_2}m$$

so that if the internal energy per unit mass of the gas is written $e(p,\rho)$, then the equation of conservation of energy per unit mass becomes

$$\tfrac{1}{2}u_1^2+e_1+\frac{p_1}{\rho_1}=\tfrac{1}{2}u_2^2+e_2+\frac{p_2}{\rho_2}$$

where for an ideal gas $p/\rho=RT$ and $e=c_vT=(1/\gamma-1)p/\rho$, where T is the absolute temperature, c_v is the specific heat per gram at constant volume and $\gamma=c_p/c_v$, where c_p is the specific heat per gram at constant pressure.

These three conservation equations

$$\rho_1u_1=\rho_2u_2=m \qquad \text{(mass)}$$

$$p_1+\rho_1u_1^2=p_2+\rho_2u_2^2 \qquad \text{(momentum)}$$

and

$$\tfrac{1}{2}u_1^2+e_1+\frac{p_1}{\rho_1}=\tfrac{1}{2}u_2^2+e_2+\frac{p_2}{\rho_2} \qquad \text{(energy)}$$

together with the internal energy relation $e(p,\rho)$ completely define the properties of an ideal gas behind a shock wave in terms of the stationary gas ahead of it.

In an experiment the properties of the gas ahead of the shock are usually known, leaving five unknowns in the four equations, which are the shock front velocity u_1, the density of the shocked gas ρ_2, the relative flow velocity behind the shock u_2, the shocked gas pressure p_2 and its internal energy e_2. In practice the shock front velocity u_1 is measured and the other four properties may then be calculated.

Mach Number

A significant parameter in shock wave theory is the Mach number. It is a local parameter defined as the ratio of the flow velocity to the local velocity of sound. The Mach number of the shock front is therefore $M_s=u_1/c_1$, where u_1 is the velocity of the shock front propagating into a gas whose velocity of sound is c_1.

The Mach number of the gas flow behind the shock front is defined as $M_f=u/c_2$, where u is the flow velocity of the gas behind the shock front

$(u<u_1)$ and c_2 is the local velocity of sound behind the shock front. There is always an increase of temperature across the shock front, so that $c_2>c_1$ and $M_s>M_f$. The physical significance of the Mach number is seen by writing $M^2=u^2/c^2$, which indicates the ratio of the kinetic flow energy, $\frac{1}{2}u^2$ per mole, to the thermal energy, $c^2=\gamma RT$ per mole. The higher the proportion of the total gas energy to be found as kinetic energy of flow the greater is the Mach number.

Ratios of Gas Properties Across a Shock Front

A shock wave may be defined in terms of the shock Mach number M_s, the density or compression ratio across the shock front $\beta=\rho_2/\rho_1$, the temperature ratio across the shock T_2/T_1 and the compression ratio or shock strength $y=p_2/p_1$.

Given the shock strength, $y=p_2/p_1$, the conservation equations are easily solved to yield

$$M_s=\frac{u_1}{c_1}=\left(\frac{y+\alpha}{(1+\alpha)}\right)^{1/2}$$

where

$$\alpha=\frac{\gamma-1}{\gamma+1}$$

$$\beta=\frac{\rho_2}{\rho_1}=\frac{\alpha+y}{1+\alpha y}$$

and

$$\frac{T_2}{T_1}=y\left(\frac{1+\alpha y}{\alpha+y}\right)$$

Alternatively these may be written in terms of the experimentally measured parameter M_s as

$$\frac{p_2}{p_1}=y=M_s^2(1+\alpha)-\alpha$$

$$\frac{\rho_2}{\rho_1}=\beta=\frac{M_s^2}{1-\alpha+\alpha M_s^2}$$

and

$$\frac{T_2}{T_1}=\frac{[\alpha(M_s^2-1)+M_s^2][\alpha(M_s^2-1)+1]}{M_s^2}$$

For weak shocks (where p_2/p_1 is just greater than 1) β, T_2/T_1 and M_s are also just greater than unity, and the shock wave moves with the speed of sound.

Strong Shocks

The ratio $p_2/p_1\gg 1$ defines a strong shock, in which case

$$M_s^2\to\frac{(\gamma+1)}{2\gamma}y$$

and

$$\beta = \frac{\rho_2}{\rho_1} \to \left(\frac{\gamma+1}{\gamma-1}\right)$$

a limit of 6 for air and 4 for a monatomic gas for a constant γ. The flow velocity

$$u = u_1 - u_2 \to \frac{2u_1}{(\gamma+1)}$$

and the temperature ratio

$$\frac{T_2}{T_1} = \left(\frac{c_2}{c_1}\right)^2 \to \frac{(\gamma-1)}{(\gamma+1)} y$$

The temperature increase across strong shocks is of great experimental interest. The physical reason for this increase may be seen by rewriting the equation of energy conservation as $\frac{1}{2}u_1^2 + h_1 = \frac{1}{2}u_2^2 + h_2$, where $h = (e + p/\rho)$ is the total heat energy or enthalpy per unit mass. For strong shocks $h_2 \gg h_1$ of the cold stationary gas and $u_1 \gg u_2$, so that the energy equation reduces to $h_2 \approx \frac{1}{2}u_1^2$, which states that the relative kinetic energy of a stationary gas element just ahead of the shock front is converted into thermal energy when the shock wave moves over that element. The energy of the gas which has been subjected to a very strong shock wave is almost equally divided between its kinetic energy and its thermal or internal energy. This may be shown by considering the initial values of the internal energy e_1 and pressure p_1 of the cold stationary gas to be negligible quantities in the conservation equations, giving the kinetic energy per unit mass behind the shock as

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}(u_1 - u_2)^2 = e_2$$

the internal energy per unit mass of the shocked gas.

In principle, the temperature behind very strong shock waves should reach millions of degrees. In practice, real gas effects prevent this. In a monatomic gas high translational energies increase the temperature until ionization occurs and this process then absorbs energy which otherwise would increase the temperature still further. In a polyatomic gas the total energy is divided amongst the various modes (translational, rotational and vibrational) and the temperatures reached are much lower than in the case of the monatomic gas. The reduction of γ due to these processes is significant, since with increasing ionization $\gamma \to 1$, and the temperature ratio depends upon the factor $(\gamma-1)/(\gamma+1)$ which becomes very small.

(Problems 11.6, 11.7, 11.8, 11.9, 11.10, 11.11)

Problem 11.1

If the period of a pendulum with large amplitude oscillations is given by

$$T = T_0\left(1 + \tfrac{1}{4}\sin^2\frac{\theta_0}{2}\right)$$

where T_0 is the period for small amplitude oscillations and θ_0 is the oscillation amplitude, show that for θ_0 not exceeding 30°, T and T_0 differ by only 2% and for $\theta_0 = 90°$ the difference is 18%.

Problem 11.2

The equation of motion of a free undamped non-linear oscillator is given by

$$m\ddot{x} = -f(x)$$

Show that for an amplitude x_0 its period

$$\tau_0 = 4\sqrt{\frac{m}{2}}\int_0^{x_0} \frac{dx}{[F(x_0) - F(x)]^{1/2}}, \quad \text{where } F(x_0) = \int_0^{x_0} f(x)\,dx$$

Problem 11.3

The equation of motion of a forced undamped non-linear oscillator of unit mass is given by

$$\ddot{x} + s(x) = F_0 \cos \omega t$$

Writing $s(x) = s_1 x + s_3 x^3$, where s_1 and s_3 are constant, choose the variable $\omega t = \phi$, and for $s_3 \ll s_1$ assume a solution

$$x = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n}{3}\phi + b_n \sin\frac{n}{3}\phi\right)$$

to show that all the sine terms and the even numbered cosine terms are zero, leaving the fundamental frequency term and its third harmonic as the significant terms in the solution.

Problem 11.4

If the mutual interionic potential in a crystal is given by

$$V = -V_0\left[2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right]$$

where r_0 is the equilibrium value of the ion separation r, show by expanding V about V_0 that the ions have small harmonic oscillations at a frequency given by $\omega^2 \approx 72\ V_0/mr_0^2$, where m is the reduced mass.

Problem 11.5

The potential energy of an oscillator is given by

$$V(x) = \tfrac{1}{2}kx^2 - \tfrac{1}{3}ax^3$$

where a is positive and $\ll k$.

Assume a solution $x = A\cos\omega t + B\sin 2\omega t + x_1$ to show that this is a good approximation at $\omega_0^2 = \omega^2 = k/m$ if $x_1 = \alpha A^2/2\omega_0^2$ and $B = -\alpha A^2/6\omega_0^2$, where $\alpha = a/m$.

Problem 11.6

The properties of a stationary gas at temperature T_0 in a large reservoir are defined by c_0, the velocity of sound, $h_0 = c_p T_0$, the enthalpy per unit mass, and γ, the constant value of the specific heat ratio. If a ruptured diaphragm allows the gas to flow along a tube with velocity u, use the equation of conservation of energy to prove that

$$\frac{c_0^2}{\gamma-1}=\frac{\gamma+1}{2(\gamma-1)}c^{*2}$$

where c^* is the velocity at which the flow velocity equals the local sound velocity.
Hence show that if $u_1/c^*=M^*$ and $u_1/c_1=M_s$, then

$$M^{*2}=\frac{(\gamma+1)M_s^2}{(\gamma-1)M_s^2+2}$$

Problem 11.7

Using a coordinate system which moves with a shock front of velocity u_1, show from the conservation equations that c^* in problem 11.6 is given by

$$c^{*2}=u_1u_2$$

where u_2 is the relative flow velocity behind the shock front.

Problem 11.8

Use the conservation equations to prove that the pressure ratio across a shock front in a gas of constant γ is given by

$$\frac{p_2}{p_1}=\frac{\beta-\alpha}{1-\beta\alpha}$$

where $\beta=\rho_2/\rho_1$, the density ratio, and $\alpha=(\gamma-1)/(\gamma+1)$.

Problem 11.9

Use the results of problems 11.6 and 11.7 with the equation of momentum conservation to prove that the shock front Mach number is given by

$$M_s=\frac{u_1}{c_1}=\sqrt{\frac{y+\alpha}{1+\alpha}}$$

where $y=p_2/p_1$, the pressure ratio across the shock and $\alpha=\gamma-1/\gamma+1$. Hence show that the flow velocity behind the shock is given by

$$u=\frac{c_1(1-\alpha)(y-1)}{\sqrt{(1+\alpha)(y+\alpha)}}$$

Problem 11.10

The diagrams (p. 377) show (a) a shock wave of pressure p_2 and flow velocity u propagating into a stationary gas, pressure p_1, and (b) after reflexion at a rigid wall the reflected wave of pressure p_3 moving back into the gas behind the incident shock still at pressure p_2. Use the result at the end of problem 11.9 to show that the flow velocity u_r behind the reflected wave is given by

$$\frac{u_r}{c_2}=\frac{(1-\alpha)(p_3/p_2-1)}{\sqrt{(1+\alpha)(p_3/p_2+\alpha)}}$$

and since $u+u_r=0$ at the rigid wall, use this result together with the ratio for $c_2/c_1=(T_2/T_1)^{1/2}$ to prove that

$$\frac{p_3}{p_2}=\frac{(2\alpha+1)y-\alpha}{\alpha y+1}$$

where $y=p_2/p_1$ and $\alpha=(\gamma-1)/(\gamma+1)$.

Problem 11.11

Use problems 11.10 to prove that the ratio

$$\frac{p_3-p_1}{p_2-p_1}\to 2+\frac{1}{\alpha}$$

in the limit of very strong shocks. (Note that this value is 8 for $\gamma=1\cdot4$ and 6 for $\gamma=5/3$, compared with the normal acoustic pressure jump of 2 upon reflexion.)